On the Applications of Axial Representation of Trigonometric Functions

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Axial view of trigonometric functions (Siadat, 2002) provides for a geometric representation of the magnitudes and signs of these functions. It helps students develop an intuitive sense of the functions even in extreme, undefined cases. There are, however, other benefits, particularly in the realm of applications. One such case is in trigonometric identities. We can prove practically all single angle identities using this pictorial representation in the unit circle below (Figure 1).

For instance by using triangles OPM, OQN, and OJK, we can easily prove the Pythagorean identities: $\sin^2\theta+\cos^2\theta = 1$, $1+\tan^2\theta = \sec^2\theta$, and $1+\cot^2\theta = \csc^2\theta$. Similarly, using the similiarity of triangles OPM and OQN; OLP and OJK, and also, OQN and OJK, one can immediately prove the quotient identities $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ and $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$ and the reciprocal identities $\dfrac{1}{\cos\theta} = \sec\theta$, $\dfrac{1}{\sin\theta} = \csc\theta$, and $\dfrac{1}{\tan\theta} = \cot\theta$. The diagram in Figure 1 also helps with proving the cofunction identities, such as $\sin\left(\dfrac{\pi}{2}-\theta\right) = \cos\theta$ and the negative angle identities like $\sin(-\theta) = -\sin\theta$.

Another, and perhaps, more interesting, application of axial representation is in calculus—finding the derivatives of trigonometric functions. Traditionally, as in Finney and Giordano (2001), to prove that the derivative of $\sin\theta$ is $\cos\theta$, one has to resort to bounding the area of the sector of a unit circle subtended by the central angle θ, then using the "squeeze theorem" to find that the $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, followed by showing that $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\theta} = 0$. The proof will then be complete via a trigonometric

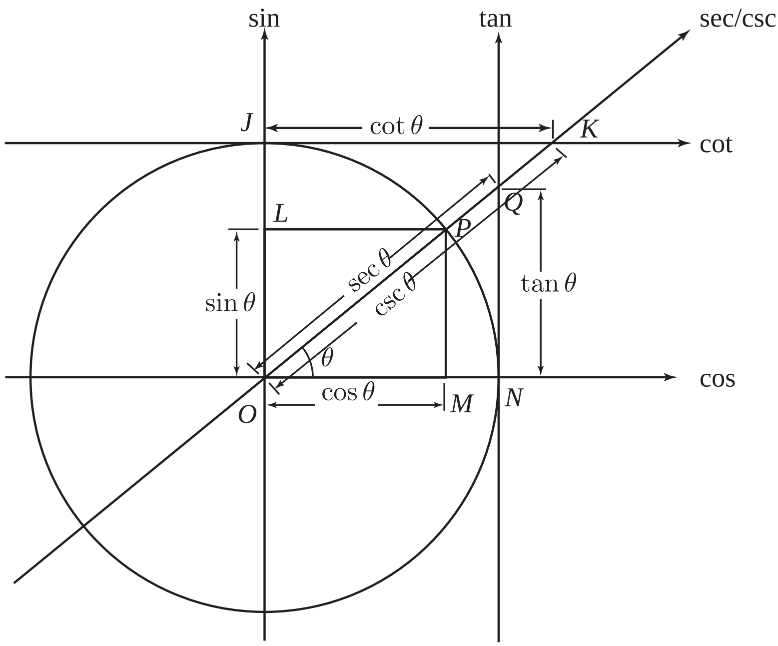

Figure 1

sum identity. We will show an intuitive alternative approach which is based on our axial model.

Consider Figure 2 which displays the central angle θ, extended by an increment η radians in a unit circle centered at O, where AD and CE are parallel to the vertical axis and BC is parallel to the horizontal axis. Also, s is the length of the arc facing η. As η becomes small, so does the arc length s, making, in the limit, the chord AC, essentially, a tangent to the circle at the point C. This makes AC perpendicular to the radius OC. As a result, triangles OCE and ABC become similar triangles and $\frac{\overline{AB}}{\overline{AC}} = \frac{\overline{OE}}{\overline{OC}} = \frac{\overline{OE}}{r} = \overline{OE}$. Now,

$$\begin{aligned}(\sin\theta)' &= \lim_{\eta\to 0} \frac{\sin(\theta+\eta) - \sin\theta}{\eta} \\ &= \lim_{s\to 0} \frac{\overline{AD} - \overline{CE}}{s} \\ &= \lim_{\overline{AC}\to 0} \frac{\overline{AB}}{\overline{AC}} = \overline{OE} = \cos\theta\end{aligned}$$

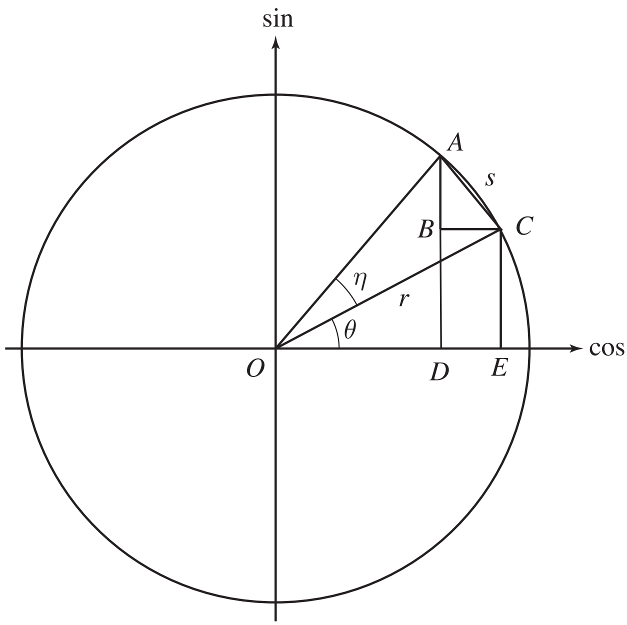

Figure 2

Using a similar argument, we can show that the derivative of $\cos\theta$ is $-\sin\theta$. Again as η gets small, so does s making, in the limit, the chord AC tangent to the circle at C. Consequently, triangles OCE and ABC become similar and we will have that $\dfrac{\overline{BC}}{\overline{AC}} = \dfrac{\overline{CE}}{\overline{OC}} = \dfrac{\overline{CE}}{r} = \overline{CE}$.

Hence,

$$
\begin{aligned}
(\cos\theta)' &= \lim_{\eta\to 0} \frac{\cos(\theta+\eta) - \cos\theta}{\eta} \\
&= \lim_{s\to 0} \frac{\overline{OD} - \overline{OE}}{s} \\
&= -\lim_{\overline{AC}\to 0} \frac{\overline{BC}}{\overline{AC}} = -\overline{CE} = -\sin\theta
\end{aligned}
$$

For θ in the third quadrant we can use the extension of the same diagram to obtain identical results.

For θ in the second and the fourth quadrants, we use a slight modification to our schematic (see Figure 3).

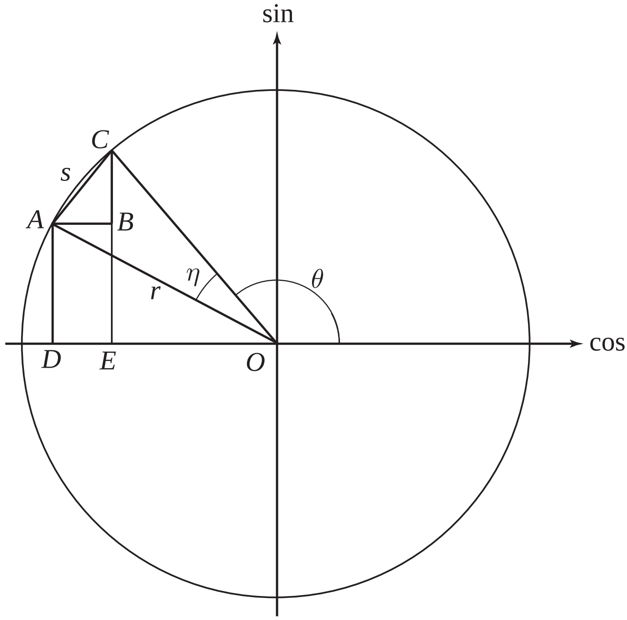

Figure 3

Here, as before, AD and CE are parallel to the vertical axis, but now AB is parallel to the horizontal axis.

Also, s is the length of the arc facing η. Again, as η becomes small, so does the arc length s, making, in the limit, the chord AC a tangent to the circle at the point C. This makes AC perpendicular to the radius OC. As a result, triangles OCE and ABC become similar triangles and $\dfrac{\overline{BC}}{\overline{AC}} = \dfrac{\overline{OE}}{\overline{OC}} = \dfrac{\overline{OE}}{r} = \overline{OE}$.

Hence,

$$\begin{aligned}(\sin\theta)' &= \lim_{\eta\to 0}\frac{\sin(\theta+\eta)-\sin\theta}{\eta}\\ &= \lim_{s\to 0}\frac{\overline{AD}-\overline{CE}}{s}\\ &= -\lim_{\overline{AC}\to 0}\frac{\overline{BC}}{\overline{AC}} = -\overline{OE} = \cos\theta\end{aligned}$$

Finally, by the similarity of triangles OCE and ABC, $\dfrac{\overline{AB}}{\overline{AC}} = \dfrac{\overline{CE}}{\overline{OC}} = \dfrac{\overline{CE}}{r} = \overline{CE}$. Thus,

$$
\begin{aligned}
(\cos\theta)' &= \lim_{\eta\to 0}\frac{\cos(\theta+\eta)-\cos\theta}{\eta}\\
&= \lim_{s\to 0}\frac{-\overline{OD}-(-\overline{OE})}{s}\\
&= -\lim_{\overline{AC}\to 0}\frac{\overline{AB}}{\overline{AC}} = -\overline{CE} = -\sin\theta
\end{aligned}
$$

References

Finney, R., Weir, M. & Giordano, F. (2001). *Thomas' calculus*, (10th ed.), New York: Addison Wesley.

Siadat, M. V. (2002). Axial view of trigonometric functions, *Mathematics Magazine, 75*(5), 396–397.

Double Negatives

Timothy Mayo

"Hey, man, you know I didn't do nothing."
You mean, my friend, that you did something.
There's a double negative in your speech.
Your meaning's the opposite of what you preach.

When two negatives come together
There is a fast change in the weather.
Two negatives cannot remain.
They'll cause each other grief and pain.

You will find double "nos" in Greek,
But in your tongue they stink and reek.
In math they cannot live in peace.
On paper please give them release.

And so two negatives must part.
I mean this with all my heart.
In their place a plus appears.
They part forever, no more tears!

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