Limits, Infinity, and Beyond
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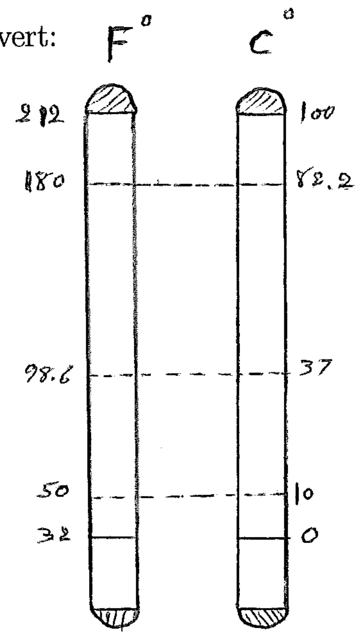
Power of Formulas

I. Fahrenheit and Celsius Degrees.

Use diagrams to convert:



Or use a formula using simple proportions:

$$\frac{C-0}{100} = \frac{F-32}{180} \qquad \text{So,} \quad C = \frac{5}{9}(F-32), \qquad \text{or} \qquad F = \frac{9}{5}C + 32$$

Example: if $F = 50°$, $C = \frac{5}{9}(50 - 32) = 10°$

Example: if $C = 100°$, $F = \frac{9}{5}(100) + 32 = 212°$

Example: if $F = 98.6°$, $C = \frac{5}{9}(98.6 - 32) = 37°$

Power of Formulas

II. Compound Interest.

Use case-by-case copmutation:

\$1000 Principal,	8% interest rate,	4 times a year compounding,	3 years.
Beginning amount:	\$1000		
First Period:	$\$1000 + \$1000(0.02) = \$1020$		
Second Period:	$\$1020 + \$1020(0.02) = \$1040.40$		
Third Period:	$\$1040.40 + \$1040.40(0.02) = \$1061.208$		

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After three years the balance is: $B = \$1268.24$

Or develop a formula using simple factoring:

$P =$ principal, $r =$ interest rate, $k =$ compounding period, $t =$ years.

Beginning Amount: P

First Period: $P + P\frac{r}{k} = P(1 + \frac{r}{k})$

Second Period: $P(1 + \frac{r}{k}) + P(1 + \frac{r}{k})\frac{r}{k} = P(1 + \frac{r}{k})(1 + \frac{r}{k}) = P(1 + \frac{r}{k})^2$

Third Period: $P(1 + \frac{r}{k})^2 + P(1 + \frac{r}{k})^2\frac{r}{k} = P(1 + \frac{r}{k})^2(1 + \frac{r}{k}) = P(1 + \frac{r}{k})^3$

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After t years the balance is: $B = P(1 + \frac{r}{k})^{kt}$

Example:: $P = \$1000$, $r = 8\%$, $k = 4$, $t = 3$ years

The balance is: $B = 1000(1 + \frac{0.08}{4})^{4(3)} = \$1268.24$

Example:: $P = \$1000$, $r = 8\%$, $k = 4$, $t = 10$ years

The balance is: $B = 1000(1 + \frac{0.08}{4})^{4(10)} = \$2208.04$

Example:: $P = \$2500$, $r = 6.5\%$, $k = 12$, $t = 20$ years

The balance is: $B = 2500(1 + \frac{0.065}{12})^{12(20)} = \$9141.17$

Secrets of Arithmetic Sequences

Finite Sums (Gauss's Story).

$1 + 2 + 3 + 4 + \dots\dots + 1000 =?$

$$\begin{array}{l} 1 \quad + \quad 2+ \quad 3+ \quad 4+ \dots\dots + 1000 \\ 1000 + 999 + 998 + 997 + \dots\dots + 1 \end{array}$$

$$S = 1000(\frac{1+1000}{2}) = 1000(500.5) = 500,500$$

More sums:

$$\begin{array}{l} 1 \quad + \quad 2+ \quad 3+ \quad 4+ \dots\dots + 1000,000 \\ 1000,000 + 999,999 + 999,998 + 999,997 + \dots\dots + 1 \end{array}$$

$$S = 1000,000(\frac{1+1000,000}{2}) = 1000,000(500,000.5) = 500,000,500,000$$

And more sums:

$$\begin{array}{l} 2 \quad + \quad 5+ \quad 8+ \quad 11+ \quad \dots\dots + 2,999 \\ 2999 + 2996 + 2993 + 2990 + \dots\dots + 2 \end{array}$$

$$S = 1000(\frac{2+2,999}{2}) = 1000(1500.5) = 1,500,500$$

So, in general:

$a_1 + a_2 + a_3 + \dots\dots + a_n$, where, $a_n - a_{n-1} = d$

$$S_n = n(\frac{a_1 + a_n}{2})$$

Secrets of Geometric Sequences

Finite Sums (King and Vazir Chess Story).

$1+2+4+8+16+ \ldots\ldots\ldots\ldots + 2^{63} = ?$

$S = 1+2+4+8+16 \ldots\ldots\ldots\ldots + 2^{63}$
$2S = 2+4+8+16+ \ldots\ldots\ldots\ldots + 2^{63} + 2^{64}$

$S - 2S = 1 - 2^{64}$,

So, $S = \dfrac{1-2^{64}}{1-2} = 2^{64} - 1 = 18,446,744,073,709,551,615$ Grains
= 4000 billion bushels of wheat (5million grains/bushel)
= 2000 years production of wheat in the world...(2 billion bushels/year)

So, in general:

$S = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1}$, where, $\dfrac{a_n}{a_{n-1}} = r$

$rS = ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1} + ar^n$

$S - rS = a - ar^n$ or $S = \dfrac{a(1-r^n)}{1-r}$

Example:

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\ldots + \frac{1}{512} = ?$ Here, $a = 1$, $r = \frac{1}{2}$.

So, $S = \dfrac{1(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} = 2(1-\dfrac{1}{1024}) = 2(\dfrac{1023}{1024}) = \dfrac{1023}{512}$

Secrets of Infinite Sequences

An Infinite Sum:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\dots\dots =?$$

$$S=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\dots\dots =?$$

$S=$ infinite number of $\frac{1}{2}'s=\infty$

Another Infinite Sum:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\dots\dots =?$$

$$S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\dots\dots =?$$

This is an infinite sum of a geometric sequence with $a=1$, and $r=\frac{1}{2}$:

We had $S=\frac{a(1-r^n)}{1-r}$ for the finite sum. But if $|r|<1$, $r^n \longrightarrow 0$, as $n \longrightarrow \infty$. Therefore, in such cases, $S=\frac{a}{1-r}$ and so, $S=\frac{1}{1-\frac{1}{2}}=2$ (for our infinite geometric sequence).

And if the signs alternate:

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\dots\dots$$

$a=1,\ r=-\frac{1}{2}$, and $\qquad S=\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{2})}=\frac{2}{3}$

Now for a different example:

$$3+\frac{3}{4}+\frac{3}{16}+\frac{3}{64}+\dots\dots$$

$a=3,\ r=\frac{1}{4}$, and $\qquad S=\frac{a}{1-r}=\frac{3}{1-\frac{1}{4}}=4$

Applications to Infinite Repeating Decimals

Example: 0.33333333333333333333333333....................... =?

$$S = 0.3 + 0.03 + 0.003 + 0.0003.................., \text{ where, } a = 0.3 \text{ and } r = 0.1$$

So, $$S = \frac{a}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$$

Another Example: 0.25252525................................. =?

$$S = 0.25 + 0.0025 + .000025 + ..., ..\text{where, } a = 0.25 \text{ and } r = 0.01$$

So, $$S = \frac{a}{1-r} = \frac{0.25}{1-0.01} = \frac{0.25}{0.99} = \frac{25}{99}$$

Still another example: 1.12121212................................. =?

$$S = 1 + 0.12 + 0.0012 + .000012 + ...\text{where, } a = 0.12 \text{ and } r = 0.01$$

So, $$S = 1 + \frac{a}{1-r} = 1 + \frac{0.12}{1-0.01} = 1 + \frac{12}{99} = 1 + \frac{4}{33} = \frac{37}{33}$$

And another example: 45.234234234234................................. =?

$$S = 45 + 0.234 + 0.000234 + 0.000000234 + ...\text{where, } a = 0.234 \text{ and } r = 0.001$$

So, $$S = 45 + \frac{a}{1-r} = 45 + \frac{0.234}{1-0.001} = 45 + \frac{234}{999} = 45 + \frac{26}{111} = \frac{5021}{111}$$

Comaring Infinities
∞ V.S. ∞

Question: Which is larger, the number of odd integers or the even integers?

$$
\begin{array}{ccccccccc}
1 & 3 & 5 & 7 & 9 & 11 & 13 \ldots\ldots\ldots\ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \ldots\ldots\ldots\ldots \\
2 & 4 & 6 & 8 & 10 & 12 & 14 \ldots\ldots\ldots\ldots
\end{array}
$$

So, the infinity of odd integers equals the infinty of even integers. This infinity is denoted by $\aleph_o$ (aleph null), which is the first transfinite number.

Another question: Which is larger, the number of all integers or the number of even integers?

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \ldots\ldots\ldots\ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \ldots\ldots\ldots\ldots \\
2 & 4 & 6 & 8 & 10 & 12 & 14 \ldots\ldots\ldots\ldots
\end{array}
$$

So, the infinity of even integers, $\aleph_o$, equals the infinty of all integers, $\aleph_o$.

When it comes to infinity, a part can be equal to the whole.

Still another question: Which is larger, the number of fractions or the number integers?

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \ldots\ldots\ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \ldots\ldots\ldots \\
\frac{1}{1} & \frac{2}{1} & \frac{1}{2} & \frac{3}{1} & \frac{1}{3} & \frac{2}{2} & \frac{4}{1} & \frac{1}{4} & \frac{2}{3} & \frac{3}{2} \ldots\ldots\ldots
\end{array}
$$

So, the infinity of rational numbers, $\aleph_o$, equals the infinty of all integers, $\aleph_o$.

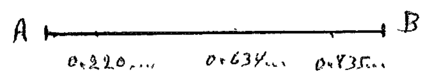
Operations on Infinities

Can we fill an already full hotel with infinite number of people? (Mr.Hilbert's Infinity Hotel).

So, $1 + \aleph_o = \aleph_o$, $2 + \aleph_o = \aleph_o$, $3 + \aleph_o = \aleph_o$, ..., $\aleph_o + \aleph_o = \aleph_o$, $\aleph_o \cdot \aleph_o = \aleph_o^2 = \aleph_o$

Question: Are the number of points on a line equal to the number of natural numbers?

Suppose we have a line segment one inch long, and suppose we have a way to put all irrational numbers, i.e., those infinite non-repeating decimals on this line, into a one to one corresesopondence with all natural numbers as follows:



$1 \leftrightarrow 0.4568920$..
$2 \leftrightarrow 0.2204987$..
$3 \leftrightarrow 0.5123479$..
$4 \leftrightarrow 0.9823765$..
$5 \leftrightarrow 0.6347801$..
$6 \leftrightarrow 0.7890353$..
$7 \leftrightarrow 0.8351254$..
....
....
....

Now, we can construct a number which is not in this list.

Take, for example, 0.5438921..
which is not in the list and for which there is no one to one correspondence with natural numbers.

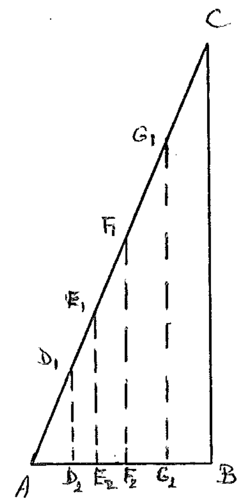
So, the infinity of real numbers is larger than the infinity of natural numbers.

We call $\aleph_1$ = infinity of all points on a line one inch long. So, $\aleph_1 > \aleph_o$

Comaring Infinities (Revisited)
∞ V.S. ∞

Question: Which is larger, the infinity of all the points on a line one inch long, or the infinitiy of all the points on a line one mile long? In other words, are there as many points one a line one inch long as there are on a line one mile long?

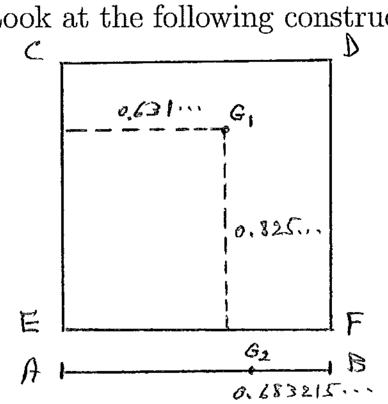
The answer is YES. Look at the following construction.



So, there is a one to one correspondence between the points on a line one inch long and the points on a line one mile long, or any other line of any length.

Another question: Which is larger, the infinity of all the points on a line or the infinitiy of all the points on a plane? In other words, are there as many points on a line as there are on a plane?

The answer is YES. Look at the following construction.

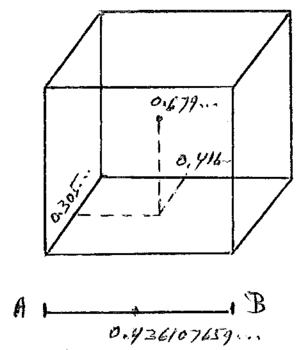


To prove the one to one correspomdence just break the decimal into two parts (odd and even) and use the two parts as the coordinates of a point in the square.

So, there is a one to one correspondence between the points on a line and the points on a plane.

Another question: Which is larger, the infinity of all the points on a line or the infinitiy of all the points witin a cube? In other words, are there as many points one a line as there are in a cube?

The answer is YES. Look at the following construction.



To prove the one to one correspomdence just break the decimal into three parts and use the three parts as the coordinates of a point in the cube.

There is still a larger infinity than the infinity of natural numbers, or the infinity of all the points on a line, or within a square or cube. This is the infinty of all possible curves, including the ones with the most unusual shapes. This infinity is denoted by $\aleph_2$. So, $\aleph_2 > \aleph_1 > \aleph_o$

Canor-Kronecker Dispute:

Refuting Cantor's work, Kronecker said

"God made the integers, all the rest is man made."

Cantor Finally Vindicated:

At the 1897 congress of mathematicians, in Zurich, Switzerland, David Hilbert said the following in tribute to Cantor's ingenious mathematics.

"Cantor's transfinite numbers were the most astonishing product of mathematical thought, one of the most beautiful realizations of human activity in the domain of the purely intelligible........No one shall expel us from the paradise which Cantor has created for us."